

$SO(2,4)$ -covariant quantization of the Maxwell field in conformally flat spaces

Sofiane Faci*

Institute of Cosmology, Relativity and Astrophysics (ICRA - CBPF)
Rua Dr. Xavier Sigaud, 150, CEP 22290-180, Rio de Janeiro, Brazil
 (Dated: March 12, 2013)

We present an $SO(2,4)$ -covariant quantization of the free electromagnetic field in conformally flat spaces (CFS). A CFS is realized in a six-dimensional space as an intersection of the null cone with a given surface. The smooth move of the latter is equivalent to perform a Weyl rescaling. This allows to transport the $SO(2,4)$ -invariant quantum structure of the Maxwell field from Minkowski space to any CFS. Calculations are simplified and the CFS Wightman two-point functions are given in terms of their Minkowskian counterparts. The difficulty due to gauge freedom is surpassed by introducing two auxiliary fields and using the Gupta-Bleuler quantization scheme. The quantum structure is given by a vacuum state and creators/annihilators acting on some Hilbert space. In practice, only the Hilbert space changes under Weyl rescalings. Also the quantum $SO(2,4)$ -invariant free Maxwell field does not distinguish between two CFSs.

PACS numbers: 11.25.Hf, 04.62.+v, 03.65.Sq

I. INTRODUCTION

The $SO(2,4)$ -invariance of the Maxwell equation was discovered by Cunningham and Bateman a century ago. However in order to quantize the Maxwell field and due to gauge freedom, a gauge fixing condition is necessary. The Lorenz gauge is usually used, which breaks the $SO(2,4)$ invariance. Nonetheless since such a symmetry might appear to lack physical meaning, its breaking does not bother many people [1]. The purpose of the present paper is to demonstrate the benefits of keeping this fundamental symmetry when quantizing the Maxwell field in conformally flat spaces (CFS).

The starting point is the following. A classical $SO(2,4)$ -invariant field cannot, at least locally, distinguish between two CFSs [2–5]. So why not maintain the $SO(2,4)$ -invariance during the quantization process in a CFS? Doing so, a free field living in a CFS might behave like in a flat space and the corresponding Wightman two-point functions can be related to their Minkowskian counterparts. The work [6] confirms this assertion in the special case of Maxwell field in de Sitter space. Indeed, a new and simple two-point Wightman function $\langle A_\mu(x)A_{\nu'}(x') \rangle$ was found and which has the same physical (gauge independent) content as the two-point function of Allen and Jacobson [7]. This is because the Faraday propagator $\langle F_{\mu\nu}(x)F_{\mu'\nu'}(x') \rangle$ is the same.

The present work extends to general CFSs and clarify the quantum structure of the formalism developed in [6]. We use Dirac's six-cone formalism and realize all CFSs as intersections of the null cone and a given surface in a six-dimensional Lorentzian space. The introduction of auxiliary fields and the use of the Gupta-Bleuler quantization scheme are necessary to deal with gauge freedom of the Maxwell field. Another important ingredient is

the use of a well-suited coordinate system. This allows to $SO(2,4)$ -invariant CFS formulas to get a Minkowskian form. The main result is a set of Wightman two-point functions for Maxwell and auxiliary fields.

This paper is organized as follows. Sec. II sets the coordinates systems and the geometrical construction of CFSs. Sec. III defines the fields and gives their dynamical equations. In Sec. IV, the dynamical system is solved, the quantum field is explicitly constructed and the two-point functions are written down. Some technical details are given in Appdx. A. The infinitesimal $SO(2,4)$ action on the fields A_i is expanded in Appdx. B and their $SO(2,4)$ -invariant scalar product is given in Appdx. C.

II. FROM \mathbb{R}^6 TO A FOUR-DIMENSIONAL CFS

The six-dimensional Lorentzian space \mathbb{R}^6 is provided with the natural orthogonal coordinates $y^\alpha = \{y^\mu, y^4, y^5\}$ and equipped with the metric $\tilde{\eta}_{\alpha\beta} = (+ - - - +)$. Quantities related to \mathbb{R}^6 and its null cone \mathcal{C} are labeled with a tilde. We define a second coordinate system $x^I = \{x^c, x^\mu, x^+\}$,

$$\begin{aligned} x^c &= \frac{y_\alpha y^\alpha}{(y^4 + y^5)^2} & y^\mu &= x^+ x^\mu, \\ x^\mu &= 2 \frac{y^\mu}{y^4 + y^5} & y^4 &= x^+ (1 - x^c + \frac{x^2}{4}) \\ x^+ &= \frac{1}{2}(y^4 + y^5) & y^5 &= x^+ (1 + x^c - \frac{x^2}{4}), \end{aligned} \quad (1)$$

where the four components x^μ is the so-called polyspherical coordinate system [1] and $x^2 = \eta_{\mu\nu} x^\mu x^\nu$. A straightforward calculation yields

$$y^\alpha \tilde{\partial}_\alpha = x^+ \partial_+, \quad (2)$$

which means that the component x^+ carries alone the homogeneity of the y 's. Using the system x^I , the null cone reads

$$\mathcal{C} = \{x^c = 0\}. \quad (3)$$

*Electronic address: sofiane@cbpf.br

A five-dimensional surface in \mathbb{R}^6 is defined through

$$\mathcal{P}_{\tilde{K}} = \left\{ x^+ = \tilde{K} \right\}, \quad (4)$$

where the real and smooth function $\tilde{K} := \tilde{K}(x^\mu, x^c)$ depends only on x^μ and x^c and is then homogeneous of degree 0. The intersection of \mathcal{C} and $\mathcal{P}_{\tilde{K}}$ is a four-dimensional space

$$X_K = \mathcal{C} \cap \mathcal{P}_{\tilde{K}}. \quad (5)$$

where the index K in X_K refers to $K = \tilde{K}(x^\mu, x^c = 0)$.

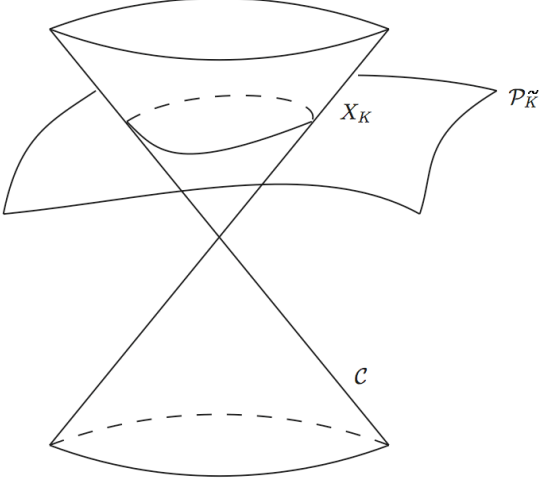


FIG. 1: The intersection of the null cone \mathcal{C} with a five-dimensional surface $\mathcal{P}_{\tilde{K}}$ produces a four-dimensional conformally flat space X_K .

Regarding to its metric inherited from $\tilde{\eta}_{\alpha\beta}$, precisely

$$ds_K^2 = dy^\alpha dy_\alpha \Big|_{\substack{x^c=0 \\ x^+=\tilde{K}}} = K^2(x) \eta^{\mu\nu} dx_\mu dx_\nu, \quad (6)$$

X_K turns out to be a CFS. A smooth move of the surface $\mathcal{P}_{\tilde{K}}$, which corresponds to changing the function \tilde{K} , amounts to perform a Weyl rescaling. This locally relates all CFSs and permits to go from one to another. Note that for $\tilde{K} = 1$, X_K reduces to Minkowski space X_M and accordingly the four components system x^μ yields the usual cartesian system. The gradients

$$\tilde{\Upsilon}_I = \frac{\partial}{\partial x_K^I} \ln \tilde{K}, \quad \Upsilon_I = \tilde{\Upsilon}_I \Big|_{x^c=0}, \quad (7)$$

are extensively used in this article. The function \tilde{K} does not depend on x^+ and thus $\Upsilon_+ = \tilde{\Upsilon}_+ = 0$.

The choice of the function \tilde{K} , including its x^c dependence has to be done in such a way to ensure the invariance (in \mathbb{R}^6) of the surface $\mathcal{P}_{\tilde{K}}$ under the action of

the isometry group associated to the desired X_K four-dimensional space. Since the \mathbb{R}^6 null-cone is $SO(2,4)$ -invariant, the resulting X_K will be invariant under its isometry group. Let us consider an example: $\tilde{K} = [1 + H^2(x^c - \frac{x^2}{4})]^{-1}$, where H^2 is a constant. The associated surface $\mathcal{P}_{\tilde{K}}$ and thus the corresponding X_K are left invariant under the action of de Sitter group [11]. Also, X_K is a de Sitter space.

III. THE MAXWELL FIELD FROM \mathbb{R}^6

In this section, we explain how to obtain the $SO(2,4)$ -invariant Maxwell field in X_K from a six-dimensional one-form. Following Dirac [10], we consider a one-form \tilde{a} defined in \mathbb{R}^6 homogeneous of degree $r = 0$ and which decomposes on the $\{dy^\alpha\}$ basis as

$$\tilde{a} = \tilde{a}_\alpha dy^\alpha. \quad (8)$$

The components \tilde{a}_α are homogeneous of degree $r = -1$ and obey to the equation

$$\square_6 \tilde{a}_\alpha = \tilde{\eta}^{\beta\gamma} \frac{\partial}{\partial y^\beta} \frac{\partial}{\partial y^\gamma} \tilde{a}_\alpha = 0. \quad (9)$$

This equation is naturally invariant under the $SO(2,4)$ action since this group has a linear action when acting in \mathbb{R}^6 . We then decompose the one-form \tilde{a} on the basis $\{dx\}$ corresponding to the system x^I (1), with a slight but capital modification on the dx^+ component. There are two ways, the first decomposition reads

$$\tilde{a} = \tilde{A}_c^M dx^c + \tilde{A}_\mu^M dx^\mu + \tilde{A}_+^M \frac{dx^+}{x^+}. \quad (10)$$

The second is given by

$$\begin{aligned} \tilde{a} &= \tilde{A}_c^K dx^c + \tilde{A}_\mu^K dx^\mu + \tilde{A}_+^K \frac{\tilde{K}}{x^+} d\left(\frac{x^+}{\tilde{K}}\right), \\ &= (\tilde{A}_c^K - \tilde{\Upsilon}_c \tilde{A}_+^K) dx^c + (\tilde{A}_\mu^K - \tilde{\Upsilon}_\mu \tilde{A}_+^K) dx^\mu + \tilde{A}_+^K \frac{dx^+}{x^+}. \end{aligned} \quad (11)$$

Now, identifying (10) with (11), one obtains the relation between the fields \tilde{A}_I^K and \tilde{A}_I^M through

$$\tilde{A}_I^K = \tilde{W}_I^J \tilde{A}_I^M = \tilde{A}_I^M + \tilde{\Upsilon}_I \tilde{A}_+^M. \quad (12)$$

All the fields \tilde{A}_I^K and \tilde{A}_I^M are by construction homogeneous of degree $r + 1 = 0$. As a consequence, $\tilde{A}_I^K = A_I^K$ and $\tilde{A}_I^M = A_I^M$. This amounts to project the fields \tilde{A}_I^K on \mathcal{P}_K and \tilde{A}_I^M on \mathcal{P}_M . Then projecting the fields on the null cone \mathcal{C} yields

$$A_I^K = W_I^J A_I^M = A_I^M + \Upsilon_I A_+^M. \quad (13)$$

Thus A_I^K and A_I^M are respectively X_K and Minkowski fields. Though in a slightly different maner, this relation

was obtained in [6] in the particular case of de Sitter space and was called the “extended Weyl transformation”. The fields A_+ and A_c are auxiliary fields and the field A_μ is, up to the condition $A_+ = 0$, the Maxwell field. This will become clear here after.

Let us now turn to the dynamical equations. Our strategy is to transport Minkowskian $SO(2, 4)$ -invariant equations to get $SO(2, 4)$ -invariant equations in the X_K space. The first step is thus to write down the Minkowskian equations which are obtained using the equation (9) and the relation (A3). This system reads

$$\begin{cases} \partial^2 A_\mu^M + \partial_\mu A_c^M = 0 \\ \partial A^M + A_c^M = \frac{1}{2} \partial^2 A_+^M \\ \partial^2 A_c^M = 0. \end{cases} \quad (14)$$

The corresponding system in X_K is obtained using (13),

$$\begin{cases} \partial^2 A_\mu^K + \partial_\mu A_c^K = \partial^2 (\Upsilon_\mu A_+^K) + \partial_\mu (\Upsilon_c A_+^K) \\ \partial A^K + A_c^K = \left(\Upsilon_c + \partial \Upsilon + \frac{1}{2} (\partial^2 + 2\Upsilon \partial) \right) A_+^K \\ \partial^2 A_c^K = \partial^2 (\Upsilon_c A_+^K), \end{cases} \quad (15)$$

where all contractions are performed using $\eta_{\mu\nu}$ even though we are in the curved space X_K . The field A_μ obeying to the system above is not yet the Maxwell one. Nevertheless, the constraint $A_+^K = 0$, simplifies the system (15) and leads to

$$\begin{cases} \partial^2 A_\mu^K - \partial_\mu \partial A^K = 0 \\ \partial^2 \partial A^K = 0. \end{cases} \quad (16)$$

Despite their Minkowskian form, these equations are the Maxwell equation and a conformal gauge condition on any conformally flat space. This is due to the use of the polyspherical coordinate system (1), which makes apparent the flatness feature of the X_K spaces. The constraint $A_+ = 0$ reduces the extended Weyl transformation (13) into the identity

$$A_I^K = A_I^M, \quad (17)$$

recovering the ordinary vanishing conformal weight of the Maxwell field A_μ . After some algebra, the covariant form of (16) takes the form

$$\begin{cases} \square A_\mu^K - \nabla_\mu \nabla A^K + \frac{1}{4} R_{\mu\nu} A_K^\nu = 0 \\ \nabla_\mu (\nabla^\mu \nabla^\nu + S^{\mu\nu}) A_\nu^K = 0, \end{cases} \quad (18)$$

where $S_{\mu\nu} = -2(R_{\mu\nu} - \frac{1}{3} R g_{\mu\nu})$. The first line (resp. the second one) is the covariant Maxwell (resp. the Eastwood-Singer gauge [12]) equation in an arbitrary X_K space. This conformal gauge was first derived by Bayen and Flato in Minkowski space [13]. Its extension to curved spaces (even CFSs) is not trivial and can be performed using adapted tools like the Weyl-gauging technique [14] or the Weyl-to-Riemann method [15].

Note that the system (18) is valid only if $A^+ = 0$ (an $SO(2, 4)$ -invariant constraint). But the latter has to be fixed at the end of the quantization process, not at the beginning. Indeed, the auxiliary field A_+ acts as a Faddeev Popov ghost field and its retention during the quantization process is necessary. The constraint $A^+ = 0$ will be applied on the quantum space to select an invariant subspace of physical states and the Wightman functions thus include the whole big space. This is related to the undecomposable group representation (see appendix B).

IV. CANONICAL QUANTIZATION

We now apply the Gupta-Bleuler quantization scheme [16–18]. This can be summarized as follows. We have seen that A_μ^K is interpreted as the Maxwell field in the Eastwood-Singer gauge (18) on the space X_K when the constraint $A_+ = 0$ is applied. The problem is that pure gauge solutions ($A_\mu^K = \nabla_\mu \Lambda$, with $\nabla_\mu (\nabla^\mu \nabla^\nu + S^{\mu\nu}) \nabla_\nu \Lambda = 0$ and $A_+ = 0$) are orthogonal to all the solutions including themselves. As a consequence, the space of solutions is degenerate and no Wightman functions can be constructed. To fix this problem, we consider the system (15), instead of (18), for which $A_+ \neq 0$ and thus a causal reproducing kernel can be found. This means that for quantum fields \hat{A}_I acting on some Hilbert (or Krein) space \mathcal{H} , we cannot impose the operator equation $\hat{A}_+ = 0$. Instead, we define the subspace of physical states $\mathcal{H}_{phy} \in \mathcal{H}$ which cancels the action of \hat{A}^+ . Then the Maxwell equation and the Eastwood-Singer gauge hold in the mean on the space \mathcal{H}_{phy} . The task seems complicated at first sight, but thanks to the correspondence (13) we only need to solve the Minkowskian system (14), which is already done in [19]. Indeed, using the Weyl equivalence between CFSs, the whole structure of an $SO_o(2, 4)$ -covariant free field theory can be transported from Minkowski to another CFS. In the following, we solve the dynamical equations, obtain the modes, determine the quantum field, the subspace of physical states and finally compute the two-point functions.

A. The mode solutions

The solutions of the Minkowskian system (14) can be obtained from [6] and read

$$A_{lm(\alpha)I}^M(x) = \epsilon_{(\alpha)I}^M(x) \phi_{lm}^M(x), \quad (19)$$

where $\epsilon_{(\alpha)}^M(x)$ are polarization vectors whose components are given by $\epsilon_{(\alpha)I}^M(x) = S_I^\beta(x) \tilde{\eta}_{\alpha\beta}$ and verifying $\langle \epsilon_{(\alpha)}^M, \epsilon_{(\beta)}^M \rangle = -\tilde{\eta}_{\alpha\beta}$, with respect to the scalar product (C1). The matrix $S_I^\beta(x)$ relates the fields A_I^M and a_α^M (A1). The scalar modes $\phi_{lm}^M(x)$ are solutions of the Minkowskian $SO(2, 4)$ -invariant (or massless) scalar field equation $\partial^2 \phi^M = 0$,

$$\phi_{lm}^M(x) = c_l \mathcal{Y}_{lm}(x), \quad (20)$$

where $\mathcal{Y}_{lm}(x)$ denotes the usual hyperspherical harmonics. The normalization constant c_l is chosen in order to get $\langle \phi_{lm}^M, \phi_{l'm'}^M \rangle_{KG} = \delta_{ll'} \delta_{mm'}$, with respect to the Klein-Gordon scalar product. As a consequence, the solutions (19) are normalized with respect to (C1),

$$\langle A_{lm(\alpha)}^M, A_{l'm'(\beta)}^M \rangle = -\tilde{\eta}_{\alpha\beta} \delta_{ll'} \delta_{mm'}. \quad (21)$$

Thus the general solution of the system (14) reads

$$A^M(x) = \sum_{lm\alpha} a_{lm(\alpha)}^M A_{lm(\alpha)}^M(x), \quad (22)$$

where $a_{lm(\alpha)}^M$ are real constants.

Let us now turn to the modes of the system (15). They are obtained thanks to the extended Weyl transformation (13) applied on the Minkowskian modes (19)

$$A_{lm(\alpha)I}^K(x) = A_{lm(\alpha)I}^M(x) + \Upsilon_I(x) A_{lm(\alpha)+}^M(x). \quad (23)$$

These modes are normalized like (21) but according to the scalar product (C2). The general solution on X_K reads

$$A^K(x) = \sum_{lm\alpha} a_{lm(\alpha)}^K A_{lm(\alpha)}^K(x). \quad (24)$$

where the $a_{lm(\alpha)}^K$ are some real constants.

Note that when

$$A_+^M = A_+^K = A_+ = 0, \quad (25)$$

the solutions (22) and (24) solve the Maxwell equation in the Eastwood-Singer gauge.

B. The quantum field and physical states

We can now define the quantum fields and construct the Fock spaces as usual. The quantum fields corresponding to (22) and (24) are respectively defined through

$$\hat{A}^M(x) = \sum_{lm\alpha} A_{lm(\alpha)}^M(x) \hat{a}_{lm(\alpha)} + A_{lm(\alpha)}^{M*}(x) \hat{a}_{lm(\alpha)}^\dagger, \quad (26)$$

$$\hat{A}^K(x) = \sum_{lm\alpha} A_{lm(\alpha)}^K(x) \hat{a}_{lm(\alpha)} + A_{lm(\alpha)}^{K*}(x) \hat{a}_{lm(\alpha)}^\dagger, \quad (27)$$

where the operators $\hat{a}_{lm(\alpha)}$ and $\hat{a}_{lm(\alpha)}^\dagger$ are respectively the annihilators and creators of the modes (19) in X_M and the modes (23) in X_K . The use of the same annihilators and creators for all CFSs is highly important for our purpose. Indeed, this allows to define the the same vacuum state $|0\rangle$ through

$$\hat{a}_{lm(\alpha)} |0\rangle = 0, \quad (28)$$

for any annihilator. The one-particle states are built by applying the creators on the vacuum state

$$|A_{lm(\alpha)}\rangle = \hat{a}_{lm(\alpha)}^\dagger |0\rangle. \quad (29)$$

and the multiple particle states of the Fock spaces are constructed as usual. Moreover, the annihilation and creation operators obey to the following algebra

$$\begin{aligned} [\hat{a}_{lm(\alpha)}, \hat{a}_{l'm'(\beta)}] &= [\hat{a}_{lm(\alpha)}^\dagger, \hat{a}_{l'm'(\beta)}^\dagger] = 0 \\ [\hat{a}_{lm(\alpha)}, \hat{a}_{l'm'(\beta)}^\dagger] &= -\tilde{\eta}_{\alpha\beta} \delta_{ll'} \delta_{mm'}. \end{aligned} \quad (30)$$

The subset of physical states in both spaces is determined thanks to the classical physical solutions (22) and (24) verifying (25). In quantum language, $|A_{phy}\rangle$ is a physical state iff

$$\hat{A}_+^{(+)} |A_{phy}\rangle = 0, \quad (31)$$

where $\hat{A}_+^{(+)}$ is the annihilator part of \hat{A}_+ . This implies the equality

$$\langle A_{phy} | \hat{A}_+(x) | B_{phy} \rangle = 0, \quad (32)$$

for any physical states $|A_{phy}\rangle$ and $|B_{phy}\rangle$. Also, the subspace of physical states is the same in all CFSs, which allows to transport physical quantities from Minkowski space into the X_K space. As a consequence, one obtains

$$\begin{aligned} \langle A_{phy} | \square \hat{A}_\mu^K - \nabla_\mu \nabla \hat{A}^\mu^K + \frac{1}{4} R_{\mu\nu} \hat{A}_K^\nu | B_{phy} \rangle &= 0 \\ \langle A_{phy} | \nabla_\mu (\nabla^\mu \nabla^\nu + S^{\mu\nu}) \hat{A}_\nu^K | B_{phy} \rangle &= 0, \end{aligned} \quad (33)$$

in X_K and the corresponding minkowskian system in X_M . The quantum fields fulfill the Maxwell equation together with the Eastwood-Singer gauge in the mean on the physical states.

C. Two-point functions

We show in this part how to get the Wightman two-point functions on X_K from their Minkowskian counterparts. The Wightman functions related to the Minkowskian fields A_I^M are defined through

$$\begin{aligned} D_{IJ'}^M(x, x') &= \langle 0 | \hat{A}_I^M(x) \hat{A}_{J'}^M(x') | 0 \rangle, \\ &= \sum_{lm\alpha} A_{lm(\alpha)I}^M(x) A_{lm(\alpha)J'}^M(x'). \end{aligned} \quad (34)$$

Their expressions are given in [19] and read

$$\begin{cases} D_{\mu\nu'}^M(x, x') &= +\eta_{\mu\nu'} D_M^s(x, x') \\ D_{\mu+}^M(x, x') &= -(x - x')_\mu D_M^s(x, x') \\ D_{c+}^M(x, x') &= -2 D_M^s(x, x') \\ D_{++}^M(x, x') &= +\frac{1}{8\pi^2} \\ D_{\mu c}^M(x, x') &= 0 \\ D_{cc}^M(x, x') &= 0, \end{cases} \quad (35)$$

where $D_M^s(x, x') = \frac{-1}{8\pi^2 \sigma_0}$, with $\sigma_0 = \frac{(x-x')^2}{2}$, stands for the Wightman two-point function related to the Minkowskian massless scalar field.

The Wightman two-point functions related to the field A_I^K are given by

$$\begin{aligned} D_{IJ'}^\kappa(x, x') &= \langle 0 | \hat{A}_I^\kappa(x) \hat{A}_{J'}^\kappa(x') | 0 \rangle, \\ &= \sum_{lm\alpha} A_{lm(\alpha)I}^\kappa(x) A_{lm(\alpha)J'}^\kappa(x'). \end{aligned} \quad (36)$$

Now, using (23), (34) and (36), allows to write the following capital formula

$$D_{IJ'}^\kappa(x, x') = D_{IJ'}^M(x, x') + \Theta_{IJ'}^\kappa(x, x'), \quad (37)$$

where the Θ terms read

$$\begin{aligned} \Theta_{IJ'}^\kappa(x, x') &= \Upsilon_I(x) D_{+J'}^M(x, x') \\ &\quad + \Upsilon_{J'}(x') D_{I+}^M(x, x') \\ &\quad + \Upsilon_I(x) \Upsilon_{J'}(x') D_{++}^M(x, x'). \end{aligned} \quad (38)$$

The Wightman two-point functions (37) read

$$\begin{cases} D_{\mu\nu'}^\kappa &= + \left(\eta_{\mu\nu} - \Upsilon_\mu \Upsilon_{\nu'} \sigma_0 - \Upsilon_\mu (x - x')_{\nu'} \right. \\ &\quad \left. - \Upsilon_{\nu'} (x - x')_\mu \right) D_M^s \\ D_{\mu+}^\kappa &= - \left(\Upsilon_\mu \sigma_0 + (x - x')_\mu \right) D_M^s \\ D_{c+}^\kappa &= - \left(2 + \Upsilon_c \sigma_0 \right) D_M^s \\ D_{++}^\kappa &= + \frac{1}{8\pi^2} \\ D_{\mu c}^\kappa &= - \left(\Upsilon_\mu \Upsilon'_c \sigma_0 + 2\Upsilon_\mu + \Upsilon'_c (x - x')_\mu \right) D_M^s \\ D_{cc}^\kappa &= - \left(\Upsilon_c \Upsilon'_c \sigma_0 + 2\Upsilon_c + 2\Upsilon'_c \right) D_M^s, \end{cases} \quad (39)$$

where $D_{IJ'}^\kappa \equiv D_{IJ'}^\kappa(x, x')$, $D_s^M \equiv D_s^M(x, x')$, $\Upsilon_I \equiv \Upsilon_I(x)$ and $\Upsilon'_I \equiv \Upsilon_I(x')$.

To end this paper, let us consider an important particular case, that corresponding to de Sitter space. This case is obtained by specifying

$$\tilde{K} = \frac{1}{1 + H^2(x^c - \frac{x^2}{4})},$$

where H is related to the de Sitterian Ricci scalar through $R = 12H^2$. The gradients (7) read

$$\Upsilon_\mu = \frac{H^2 K}{2} \eta_{\mu\nu} x^\nu, \quad \Upsilon_c = -\frac{H^2 K}{2}.$$

In this case we obtain simple expressions for the two-point functions related to the fields A_I^H on de Sitter space. The three more relevant yield

$$\begin{cases} D_{\mu\nu'}^H &= \frac{H^2}{8\pi^2} \left(\frac{1}{\mathcal{Z}-1} g_{\mu\nu'} - n_\mu n_{\nu'} \right) \\ D_{\mu+}^H &= \frac{1}{H} \sqrt{\mathcal{Z}^2 - 1} n_\mu \\ D_{++}^H &= \frac{1}{8\pi^2}, \end{cases} \quad (40)$$

where we have used (de Sitter is a maximally symmetric space) the standard unit tangent vectors $n_\mu(x, x')$

and $n_{\nu'}(x, x')$, the parallel propagator along the geodesic $g_{\mu\nu'}(x, x')$ and the usual function \mathcal{Z} of the geodesic distance $\mu(x, x')$ relating x and x' , $\mathcal{Z} = \cosh(H\mu)$. See [6] for a more precise statement. Note that the two-point function $D_{\mu\nu'}^H$ has the same physical content with the Allen and Jacobson two-point function [7].

V. CONCLUSION

An $SO(2, 4)$ -covariant quantization of the Maxwell field in an arbitrary conformally flat space was presented. Following Dirac's six-cone formalism, all conformally flat spaces X_K are realized as intersections of the null cone and a given surface $\mathcal{P}_{\tilde{K}}$. The quantum field was explicitly constructed using the Gupta-Bleuler canonical quantization scheme and the Wightman two-point functions were given. The price to pay for this simplicity and the maintaining of the $SO(2, 4)$ invariance during the whole quantization process was the introduction of two auxiliary fields A_c and A_+ . As a consequence, the Maxwell field A_μ does not propagate "alone" but together with its two auxiliary fields. The propagation must use all the Wightman functions (39) and not only the "purely" Maxwell one $D_{\mu\nu}$. Nonetheless, in a recent work [20], we have used the functions (40) to propagate the Maxwell field generated by two charges of opposite sign placed at the two poles of a de Sitter space. The calculations showed that only $D_{\mu\nu}$ is involved, which trivialize the problem. One can consider to use the two-point functions (39) to propagate the electromagnetic field for some charge distribution given in other CFSs, like FLRW spaces for instance.

One concludes that is much worth to maintain the $SO(2, 4)$ symmetry during the whole quantization process when dealing with Maxwell field in a conformally flat space. The problem then goes back to Minkowski and the calculations become much easier. In fact, the classical and quantum structures of the free Maxwell field are locally the same in all conformally flat spaces. The remained question is to know if this is true for other free fields and how to deal with $SO(2, 4)$ -invariant interactions?

Acknowledgments

I would like to thank M. Novello, J. Renaud and E. Huguet for illuminating discussions and the CNPq for financial support.

Appendix A: The transformations relating a_α to A_I

Considering (8) and (10), expressing the basis $\{dy\}$ in terms of $\{dx\}$ and then identifying both sides, one obtains the expression of \tilde{A}_I^M in terms of \tilde{a}_α . We find,

after using the homogeneity properties,

$$A_I^M = S_I^\beta(x) a_\alpha^M, \quad (\text{A1})$$

which reads

$$\begin{cases} A_c^M = a_5^M - a_4^M \\ A_\mu^M = a_\mu^M - \frac{1}{2}(a_5^M - a_4^M)x_\mu \\ A_+^M = a_5^M(1-x^2) + a_4^M(1+x^2) + a^M \cdot x. \end{cases} \quad (\text{A2})$$

This system can be inverted in

$$\begin{cases} a_5^M = \frac{1}{2}\{A_+^M - A^M x + A_c^M(1-x^2)\} \\ a_4^M = \frac{1}{2}\{A_+^M - A^M x - A_c^M(1+x^2)\} \\ a_\mu^0 = A_\mu^M + \frac{1}{2}A_c^M x_\mu. \end{cases} \quad (\text{A3})$$

Following the same steps as above, one obtain the matrix linking the a_α^K to the A_I^K

$$\begin{cases} A_c^K = K \left\{ a_5^K [\Upsilon_c(1-x^2) + 1] + a_4^K [\Upsilon_c(1+x^2) - 1] \right. \\ \quad \left. + \Upsilon_c a^K \cdot x \right\} \\ A_\mu^K = K \left\{ a_5^K ((1-x^2)\Upsilon_\mu - \frac{1}{2}x_\mu) \right. \\ \quad \left. + a_4^K ((1+x^2)\Upsilon_\mu + \frac{1}{2}x_\mu) + a_\nu^K (\Upsilon_\mu x^\nu + \delta_\mu^\nu) \right\} \\ A_+^K = K \left\{ a_5^K(1-x^2) + a_4^K(1+x^2) + a^K \cdot x \right\}. \end{cases} \quad (\text{A4})$$

This system can be obtained using the Minkowskian system (A2), the relation $a_\alpha^K = K^{-1}a_\alpha^M$ (which comes out from the homogeneity properties of the fields) and the extended Weyl transformations (13). This is inverted in

$$\begin{cases} a_5^K = \frac{1}{2K} \left\{ A_+^K (1 + \Upsilon \cdot x - \Upsilon_c(1-x^2)) \right. \\ \quad \left. - A^K x + A_c^K (1-x^2) \right\} \\ a_4^K = \frac{1}{2K} \left\{ A_+^K (1 + \Upsilon \cdot x + \Upsilon_c(1+x^2)) \right. \\ \quad \left. - A^K x - A_c^K (1+x^2) \right\} \\ a_\mu^K = \frac{1}{2K} \{ -A_+^K (2\Upsilon_\mu + \Upsilon_c x_\mu) + 2A_\mu^K + A_c^K x_\mu \}. \end{cases} \quad (\text{A5})$$

Appendix B: The undecomposable $SO(2,4)$ representations of the fields A_I

The $SO(2,4)$ infinitesimal action on the field A_I is given by commutators of the group generators and the field. First, we write down the infinitesimal transformations of the Minkowskian fields A_I^M which can be found

in [6] then we transport the resulting representation into X_K . For any element $e \in SO(2,4)$, the related generator is denoted by X_e^M and whose action on the field A_I^M reads

$$\begin{aligned} (X_e^M A^M)_I &= [X_e^M, A_I^M] \\ &= X_e A_I^M + (\Sigma_e)_I^J A_J^M \end{aligned} \quad (\text{B1})$$

where the first part represents the scalar action and the second the spinorial action. Setting

$$\begin{aligned} K_\mu &= 2x_\mu x \cdot \partial - x^2 \partial_\mu \\ X_{\mu\nu} &= x_\mu \partial_\nu - x_\nu \partial_\mu, \end{aligned} \quad (\text{B2})$$

the Minkowskian infinitesimal action reads

$$\begin{cases} (X_{\mu\nu}^M A^M)_c = X_{\mu\nu} A_c^M \\ (X_{\mu\nu}^M A^M)_\lambda = X_{\mu\nu} A_\lambda^M + \eta_{\lambda[\mu} \delta_{\nu]}^\tau A_\tau^M \\ (X_{\mu\nu}^M A^M)_+ = X_{\mu\nu} A_+^M, \end{cases}$$

for the rotations,

$$\begin{cases} (P_\mu^M A^M)_c = \partial_\mu A_c^M \\ (P_\mu^M A^M)_\nu = \partial_\mu A_\nu^M \\ (P_\mu^M A^M)_+ = \partial_\mu A_+^M, \end{cases}$$

for the translations;

$$\begin{cases} (K_\mu^M A^M)_c = K_\mu A_c^M + 4(x_\mu A_c^M + A_\mu^M) \\ (K_\mu^M A^M)_\nu = K_\mu A_\nu^M + 2(x_{[\mu} \delta_{\nu]}^\lambda + x^\lambda \eta_{\mu\nu}) A_\lambda^M - 2\eta_{\mu\nu} A_+^M \\ (K_\mu^M A^M)_+ = K_\mu A_+^M, \end{cases}$$

for the special conformal transformations (SCT). Finally, we have

$$\begin{cases} (D^M A^M)_c = (x\partial + 2)A_c^M \\ (D^M A^M)_\mu = (x\partial + 1)A_\mu^M \\ (D^M A^M)_+ = (x\partial + 0)A_+^M, \end{cases}$$

for the dilations. The undecomposable structure of the fields A_I^M is made clear. Under the $SO(2,4)$ action, the component A_c^M overlaps A_μ^M which in turn overlaps A_+^M . So we have the scheme

$$\begin{cases} A_c^M \rightarrow A_+^M, A_\mu^M, A_c^M, \\ A_\mu^M \rightarrow A_+^M, A_\mu^M, \\ A_+^M \rightarrow A_+^M. \end{cases} \quad (\text{B3})$$

The second step is to transport the group action from Minkowski to the X_K space using the extended Weyl transformation (13)

$$\begin{aligned} (X_e^K A^K)_I &= W_I^J (X_e^M A^M)_J \\ &= (X_e^M A^M)_I + \Upsilon_I (X_e^M A^M)_+ \\ &= X_e A_I^M + (\Sigma_e)_I^J A_J^M + \Upsilon_I X_e A_+^M \\ &= X_e (A_I^K - \Upsilon_I A_+^K) + (\Sigma_e)_I^J (A_J^K - \Upsilon_J A_+^K) \\ &\quad + \Upsilon_I X_e A_+^K \\ &= (X_e^M A^K)_I - [(X_e \Upsilon_I) + (\Sigma_e)_I^J \Upsilon_J] A_+^K \end{aligned} \quad (\text{B4})$$

where we have used $(\Sigma_e)_+^J = 0$ for all $e \in SO(2, 4)$. Also only the second part of the last line has to be computed.

The infinitesimal $SO(2, 4)$ action on A_I^K reads

$$\begin{cases} (X_{\mu\nu}^\kappa A^\kappa)_c = X_{\mu\nu} A_c^\kappa - (X_{\mu\nu} \Upsilon_c) A_+^\kappa \\ (X_{\mu\nu}^\kappa A^\kappa)_\lambda = X_{\mu\nu} A_\lambda^\kappa + \eta_{\lambda[\mu} \delta_{\nu]}^\tau A_\tau^\kappa \\ \quad - [(X_{\mu\nu} \Upsilon_\lambda) + \eta_{\lambda[\mu} \delta_{\nu]}^\tau \Upsilon_\tau] A_+^\kappa \\ (X_{\mu\nu}^\kappa A^\kappa)_+ = X_{\mu\nu} A_+^\kappa, \end{cases}$$

for the rotations,

$$\begin{cases} (P_\mu^\kappa A^\kappa)_c = \partial_\mu A_c^\kappa - (\partial_\mu \Upsilon_c) A_+^\kappa \\ (P_\mu^\kappa A^\kappa)_\nu = \partial_\mu A_\nu^\kappa - (\partial_\mu \Upsilon_\nu) A_+^\kappa \\ (P_\mu^\kappa A^\kappa)_+ = \partial_\mu A_+^\kappa. \end{cases}$$

for the translations and

$$\begin{cases} (K_\mu^\kappa A^\kappa)_c = K_\mu A_c^\kappa + 4(x_\mu A_c^\kappa + A_\mu^\kappa) \\ \quad - [(K_\mu \Upsilon_c) + 4(x_\mu \Upsilon_c + \Upsilon_\mu)] A_+^\kappa \\ (K_\mu^\kappa A^\kappa)_\nu = K_\mu A_\nu^\kappa + 2(x_{[\mu} \delta_{\nu]}^\lambda + x^\lambda \eta_{\mu\nu}) A_\lambda^\kappa - 2\eta_{\mu\nu} A_+^\kappa \\ \quad - [(K_\mu \Upsilon_\nu) + 2(x_{[\mu} \delta_{\nu]}^\lambda + x^\lambda \eta_{\mu\nu}) \Upsilon_\lambda] A_+^\kappa \\ (K_\mu^\kappa A^\kappa)_+ = K_\mu A_+^\kappa, \end{cases}$$

for the special conformal transformations. Finally, we have

$$\begin{cases} (D^\kappa A^\kappa)_c = (x\partial + 2) A_c^\kappa - [(x\partial \Upsilon_c) + 2\Upsilon_c] A_+^\kappa \\ (D^\kappa A^\kappa)_\mu = (x\partial + 1) A_\mu^\kappa - [(x\partial \Upsilon_\mu) + \Upsilon_\mu] A_+^\kappa \\ (D^\kappa A^\kappa)_+ = (x\partial + 0) A_+^\kappa, \end{cases}$$

for the dilations. The undecomposable structure of the fields A_I^K appears in a similar way than for the Minkowskian fields (B3).

Note that the constraint $A_+^M = A_+^\kappa = 0$ (25) is $SO(2, 4)$ -invariant. This is important since this constraint defines the subset of physical states.

Appendix C: The scalar product

The $SO(2, 4)$ -invariant scalar product for the Minkowskian field A_I^M reads

$$\begin{aligned} \langle A^M, B^M \rangle &= -i \int_\Sigma \sigma_M^\mu J_\mu(A^M, B^M) \\ &= -i \int_\Sigma \sigma_M^\mu \left\{ A_M^{*\nu} \overset{\leftrightarrow}{\partial}_\mu B_\nu^M + (A_\mu^{M*} B_c^M - A_c^{M*} B_\mu^M) \right. \\ &\quad \left. + \frac{1}{2} (A_+^{M*} \overset{\leftrightarrow}{\partial}_\mu B_c^M + A_c^{M*} \overset{\leftrightarrow}{\partial}_\mu B_+^M) \right\}, \end{aligned} \quad (C1)$$

where Σ is some Cauchy surface in X_M and σ_M^μ is a surface element. An important point is that this Cauchy surface is common to all the spaces X_K since they are all conformally equivalent [21].

Using (6) and (13), the scalar product for the field A_I^K is obtained from (C1) and reads

$$\begin{aligned} \langle A^K, B^K \rangle &= \langle A^M, B^M \rangle \\ &= -i \int_\Sigma \sigma_M^\mu J_\mu(W^{-1} A^K, W^{-1} B^K) \\ &= -i \int_\Sigma \sigma_K^\mu \left\{ (A_K^{*\nu} - \Upsilon^\nu A_+^*) \overset{\leftrightarrow}{\partial}_\mu (B_\nu^K - \Upsilon_\nu B_+) \right. \\ &\quad + \frac{1}{K^2} (A_\mu^{K*} - \Upsilon_\mu A_+^*) (B_c^K - \Upsilon_c A_+) \\ &\quad - \frac{1}{K^2} (A_c^{M*} - \Upsilon_c A_+^*) (B_\mu^K - \Upsilon_\mu A_+) \\ &\quad + \frac{1}{2K^2} (A_+^{M*} \overset{\leftrightarrow}{\partial}_\mu (B_c^K - \Upsilon_c A_+) \\ &\quad \left. + \frac{1}{2K^2} (A_c^{M*} - \Upsilon_c A_+^*) \overset{\leftrightarrow}{\partial}_\mu B_+^M), \right\} \end{aligned} \quad (C2)$$

where the X_K surface element is related to its Minkoskian counterpart by $\sigma_K^\mu = K^2 \sigma_M^\mu$.

-
- [1] H.A. Kastrup, *Annalen Phys.* **17**, 631 (2008).
 - [2] I. Antoniadis, P. O. Mazur, E. Mottola, arXiv:1103.4164.
 - [3] B. Zumino, *effective lagrangians and broken symmetries* (1970)
 - [4] S. Weinberg, *Phys. Rev. D* **82**, 045031 (2010)
 - [5] S. Faci, arXiv:1206.4362
 - [6] S. Faci, E. Huguet, J. Queva, and J. Renaud, *Phys. Rev. D* **80**, 124005 (2009).
 - [7] B. Allen, T. Jacobson, *Comm. Math. Phys.*, **103**, 669 (1986).
 - [8] T. Fulton, F. Rohrlich and L. Witten, *Rev. Mod. Phys.* **34**, 442 (1962).
 - [9] A. J. Keane, R. K. Barrett, *Class. Quant. Grav.* **17** (2000) 201-218. [gr-qc/9907002].
 - [10] P.A.M. Dirac, *Ann. Math.* **37**, 429 (1936).
 - [11] E. Huguet, J. Queva, and J. Renaud, *Phys. Rev. D* **73**, 084025 (2006).
 - [12] M. Eastwood and M. Singer, *Phys. Lett.* **107A**, 73 (1985).
 - [13] Bayen F. and Flato M., *J. Math. Phys.*, **17**, 1112, (1976).
 - [14] Iorio, Alfredo and O’Raifeartaigh, L. and Sachs, I. and Wiesendanger, C., *Nucl.Phys.*, **B495**, 433-450, (1997)
 - [15] S. Faci, arXiv:1212.2599.
 - [16] S.N. Gupta, *Proc. Phys. Soc. London A* **63**, 681 (1950).
 - [17] K. Bleuler, *Helv. Phys. Acta* **23**, 567 (1950).
 - [18] J-P. Gazeau, J. Renaud and M. V. Takook: *Class. Quantum Grav.* **17** (2000) 1415-1434,
 - [19] F. Bayen *et al.*, *Phys. Rev. D* **32**, 2673 (1985).
 - [20] S. Faci, E. Huguet, and J. Renaud, *Phys. Rev. D* **84**, 124050 (2011).
 - [21] A. A. Fulling, *Aspects of quantum field theory in curved space-time*, (1989).